Mock exam

Analysis on Manifolds (2021/2022)

You have 2 hours to complete the exam.
Usage of the theory and examples from the lecture notes is allowed, with the only exceptions of the results of Exercise 8.4.7 and Example 6.1.9 from the lecture notes.
Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

Exercise 1. (8 + 8 points)

Consider the set

$$M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 | x_1^2 - x_4^2 = 1,\ x_2^2 + x_3^2 = 2\}.$$

1. Show that M is a 2-dimensional smooth manifold.
2. Compute the tangent space to M at the point $(1, -1, 1, 0)$.

Exercise 2. (8 + 8 points)

Let (x, y, z) denote the standard euclidean coordinate in $\mathbb{R}^3$.
Let $\omega \in \Omega^2(\mathbb{R}^3)$ be defined by

$$\omega := yz\, dx \wedge dy \wedge dz$$

1. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(t, u, v) = (u^2 - v^2, 2uv, t)$. Compute $\phi^*(\omega)$.
2. Let $X = \frac{\partial}{\partial x}$. Compute $\mathcal{L}_X \omega$.

Exercise 3. (8 points)

Prove, expliciting all the details, that the flat map $\flat : V \to V^*$ on a normed vector space is an isomorphism.

Exercise 4. (7 + 10 + 10 points)

Let M be a smooth manifold, $\pi : T^*M \to M$ the projection on its cotangent bundle. Define for all $q \in M$ and for all $p \in T_q^*M$ the following object

$$\eta_{(q,p)} = d\pi^*_{(q,p)} p.$$

1. What are the domain and the codomain of $d\pi$? Use this to clarify what kind of object is η.
2. Consider a chart on T^*M with local coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$. Prove that locally $\eta_{(x,\xi)} = \xi_i dx^i$.
3. Use $\omega = d\eta$ to show that T^*M is orientable.

Exercise 5. (7 + 6 + 10 points)

1. Explain, in your on words, the definition of integrals on manifolds. What are the essential ingredients entering the definition? Why do we need them?
2. Let $\eta \in \Omega^1(\mathbb{R}^2) = -x^2 dx^1 + x^1 dx^2$. Is η closed? Is it exact? Justify your answer.
3. Compute $\int_{\mathbb{S}^1} \eta$.